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AN INVESTIGATION OF OPTIMALLY ROBUST STRUCTURAL DAMPING THROUGH FRACTIONAL ORDER FEEDBACK

THESIS

Mark T. Leonard
Second Lieutenant, USAF
AFIT/GA/ENY/91D-2

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The purpose of this investigation was to determine if introducing fractional order states in a feedback system was beneficial to overall system performance. Fractional order differential equations have been used in the past primarily to model viscoelastic damping in structures. This study examined the use of fractional order differential equations in formulating a control algorithm with additional degrees of freedom. The algorithm presented is best suited for active structural damping. Including the fractional order time derivatives in the state allowed some additional flexibility in choosing relevant control parameters in the system. Optimization with respect to robustness was examined to determine a solution. Many additional questions arose in this inquiry as to the applications of fractional order states in control systems.

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AN INVESTIGATION OF OPTIMALLY ROBUST STRUCTURAL DAMPING THROUGH FRACTIONAL ORDER FEEDBACK

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

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In Partial Fulfillment of the

Requirements for the Degree of

Master of Science in Astronautical Engineering

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Second Lieutenant, USAF

December 1991

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Preface

This study was initiated to examine the feasibility of applying fractional calculus to a control algorithm. The mathematical development led to fractional order states and the solution of fractional order differential equations. The pros and cons of introducing additional states in a control law were examined. The results yielded evidence that the application of fractional order control laws have merit and should continue to be investigated.

I received an abundance of help during the course of the research and while writing this thesis. I am greatly indebted to my thesis advisor and mentor, Lt Col R. Bagley, for his insight, motivation, and patience during times of need. I would also like to thank my committee members, Dr. B. Liebst and Dr. D. Khatri, for their technical assistance. A special thanks is made to my family and friends who supported me during this investigation - especially Capt J. Blank who was a springboard for many ideas. I would like to thank my wife Kimberly for her unwavering support and understanding during the many times I was withdrawn into the books. Lastly, I would like to thank my Lord Jesus Christ for *everything* - especially since He is the only one who truly understands every aspect of fractional calculus ...

Mark T. Leonard

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List of Notation

A = plant matrix

B = control matrix

C = output matrix

C_s = stiffness matrix

c = condition number

c_k = condition number of k^{th} eigenvalue

D = feedforward matrix

D^{β} = β-order fractional derivative

$\hat{D}^{\beta}$ = modified β-order fractional derivative

$E_{\beta}(z)$ = β-order Mittag-Leffler function

F = closed loop fractional order matrix

$I(t)$ = branch cut integral

J = cost function

K = gain matrix

M = mass matrix

N = smallest integer common to all fractional derivatives

n = order of the fractional order state vector

P_I = set of desired poles on complex plane

$R(r)$ = remainder polynomial

r = variable in fractional order characteristic polynomial

r_I = variable in integer order characteristic polynomial

$\underline{u}$ = input vector

$\underline{v}_k$ = right eigenvector of λ_k

X = matrix of right eigenvectors

$\underline{x}$ = state vector

$\underline{y}$ = output vector

$\underline{w}_k$ = left eigenvector of λ_k

β = $1/N$ = basis fraction of system

Γ = gamma function

γ = incomplete gamma function

λ_k = k^{th} eigenvalue

σ = complex constant

ϕ = structural mode shape

$\Psi(r)$ = psi polynomial

$\Omega(r)$ = omega polynomial

Abstract

The purpose of this investigation was to determine if introducing fractional order states in a feedback system was beneficial to overall system performance. Fractional order differential equations have been used in the past primarily to model viscoelastic damping in structures. This study examined the use of fractional order differential equations in formulating a control algorithm with additional degrees of freedom. The algorithm presented is best suited for active structural damping. Including the fractional order time derivatives in the state allowed some additional flexibility in choosing relevant control parameters in the system. Optimization with respect to robustness was examined to determine a solution. Many additional questions arose in this inquiry as to the applications of fractional order states in control systems.

AN INVESTIGATION OF OPTIMALLY ROBUST STRUCTURAL DAMPING THROUGH FRACTIONAL ORDER FEEDBACK

I. Introduction

Objective

The purpose of this investigation is to determine the effects on a system of artificially introducing additional states with the intention of harnessing them in a control algorithm. These additional states will be determined by integrating the accelerations on a system by a fractional order rather than an integer order. The nature of the solutions to the resulting fractional order differential equations will be determined. The benefits and limitations to adding these other states to the traditional mathematical model will be explored. The goal is to determine if the benefits gained from adding the fractional order states outweigh the additional complications of implementing them.

Motivation

The motivation for this investigation is current research in digital and analog fractional order integrators and differentiators. Until recently, the accuracy of fractional order integrators and differentiators was too poor for consideration in a control algorithm. As these integrators and differentiators become better, the question arises as to how this additional information can be harnessed in a

constructive way. The possibility of using fractional states in a control algorithm has been acknowledged (5:309) but not investigated. This inquiry was initiated in hopes of using the additional information from the fractional states to improve system performance.

Background (13:115-112)

The concept of arbitrary order integration and differentiation is essentially as old as traditional calculus. It was not vigorously pursued at the onset due to the lack of apparent application in light of the many of uses for traditional calculus. Joseph Liouville was the first to perform a major study on fractional calculus. He was also one of the first persons to solve differential equations using fractional calculus. G. F. Bernhard Riemann also developed a theory on fractional calculus that based its relevant definitions on a generalized Taylor series. Various other mathematicians defined fractional operations with mixed results. The contemporary definitions are attributed to Riemann and Liouville. At present, one of the primary applications of fractional calculus has been modelling viscoelastic damping in materials (3:1412-1416; 5:304-311; 12:247-275). Traditional viscoelastic material models are constrained to be functions of integer powers of the associated frequencies. Fractional calculus allows the frequency dependency to be of an arbitrary order which better models the properties of the material. The potential applications of fractional calculus have only begun to be investigated.

Organization

This report will be organized into eight chapters. The first chapter will introduce the problem and discuss its background. Chapter 2 will examine solutions of fractional order differential equations in general. Chapter 3 will apply the techniques from chapter 2 to solve the specific class of fractional differential equations employed in the ensuing control algorithm. It will also discuss some of the characteristic behavior of the solutions to this class of fractional order differential equations. Chapter 4 will develop the fractional order control law. Chapter 5 will outline the optimization of the control algorithm. The goal of the optimization will be to make the system as insensitive to unavoidable errors as possible. Chapter 6 will examine procedures to compare the results of the optimized fractional control algorithm to traditional results. Chapter 7 will contain several example problems and discuss the general implications of the specific results. The last chapter will consist of the resulting conclusions and recommendations for future investigation.

II. Solutions of Fractional Order State Equations

Fractional State Equations

A fractional order differential equation is a mathematical expression that relates a dependant variable to fractional order derivatives of itself with respect to an independent variable. The order of the differential equation is given by the highest order derivative in the equation. The extended Riemann-Liouville definition of the fractional derivative is (4:7)

$$D^{\beta}[x(t)] \equiv \frac{d}{dt}\int_{0}^{t}\frac{x(t-\tau)}{\Gamma(1-\beta)\,\tau^{\beta}}\,d\tau, \qquad 0 \le \beta \le 1 \tag{1}$$

where β is the order of the derivative. The above definition is valid for irrational and complex β although only rational numbers will be used in the formulation of the subsequent equations. It should be noted that the fractional derivative is a linear operator. If Leibnitz's rule is applied to the above equation, the result is

$$D^{\beta}[x(t)] = \hat{D}^{\beta}[x(t)] + \frac{x(0)\,t^{-\beta}}{\Gamma(1-\beta)} \tag{2}$$

where

$$\hat{D}^{\beta}[x(t)] \equiv \int_{0}^{t}\frac{\dot{x}(t-\tau)}{\Gamma(1-\beta)\,\tau^{\beta}}\,d\tau, \qquad 0 \le \beta \le 1 \tag{3}$$

The modified linear operator of Eq (3) is the Riemann Liouville fractional integral of order 1-β of the first derivative of $x(t)$ or effectively, the -β order integral of the

function (4:17). The definitions in Eqs (1) and (3) are both valid expressions for a fractional derivative. The modified operator of Eq (3) will be used by convention.

The state equations of a structure are normally written in the familiar form

$$\dot{\underline{x}} = A\underline{x} - B\underline{u} \qquad \underline{y} = C\underline{x} + D\underline{u} \tag{4}$$

where

n = (order of the system) × (number of masses)

$\underline{x}$ = state vector (n by 1)

$\underline{y}$ = output vector (p by 1)

$\underline{u}$ = control vector (m by 1)

A,B,C,D = state formulation matrices

The traditional state formulation poses the dynamics of a system as n first order equations in matrix form. The first derivative of a state is the sum of a linear combination of the other states and the control force. Bagley has formulated the fractional order state equations (5:309) in a straight forward manner:

$$\hat{D}^{\beta}(\underline{x}) = A\underline{x} + B\underline{u} \qquad \underline{y} = C\underline{x} + D\underline{u} \tag{5}$$

where

$\beta = 1/N$ = basis fraction of svstem (≤ 1)

N = smallest integer common to all fractional derivatives

n = (order of the system) x (number of masses) x N

$\underline{x}$ = state vector (n by 1)

$\underline{y}$ = output vector (p by 1)

$\underline{u}$ = control vector (m by 1)

A,B,C,D = fractional order state formulation matrices

To avoid ambiguity, further use of the integer order state equations will be subscripted by a capital I denoting integer. The n states and the matrix A are not unique for a given system in the integer or fractional formulation. Regardless, some relationships between the states can be better to work with than others.

It should be noted that an integer order system can always be written in the fractional order form. If this is done, all of the elements in the matrix A corresponding to the fractional order states will be zero. An integer order system posed in the fractional order state equations must obviously have the same solution, but the eigenvalues of the corresponding A matrices are *not* the same. An integer order system posed in the β-order fractional equations will have N times as many eigenvalues. For each integer order eigenvalue λ_I, N eigenvalues in the β-order equations will satisfy $\lambda=(\lambda_I)^{\beta}$.

The Mittag-Leffler Solution

One approach to solving a fractional order differential equation is by using the Mittag-Leffler function (5:307):

$$E_{\beta}(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \beta k)} \tag{6}$$

Notice that if β is one, the above expression is the definition of an exponential. The β-order Mittag-Leffler function has properties for β-order derivatives analogous to the exponential function for integer order derivatives. Namely,

$$\hat{D}^{\beta}[E_{\beta}(at^{\beta})] = aE_{\beta}(at^{\beta}) \tag{7}$$

The homogeneous solution of a linear β-order differential equation is a linear combination of β-order Mittag-Leffler functions. The particular solution can be found by convoluting the Mittag-Leffler functions with the input. A system can be solved using modal analysis or the Mittag-Leffler exponential matrix (5:310). The Mittag-Leffler exponential matrix is defined like a traditional exponential matrix and has analogous properties. A system of equations can be solved given the initial conditions. Bagley has shown (2:17) that the initial values of the fractional derivatives are identically zero. The complete solution resembles the traditional state solution:

$$x(t) = E_\beta(At^\beta)x(0) + \int_0^t E_\beta[A(t-\tau)^\beta]Bu(\tau)\,d\tau \qquad (8)$$

The solution technique for fractional order differential equations and integer order differential equations are similar. The following transformation on the characteristic polynomial clears the fractional exponents:

$$r_I = r^N \qquad (9)$$

where

r_I = variable in integer order characteristic polynomial

r = variable in fractional order characteristic polynomial

This change of variables effectively increase the order of the characteristic polynomial by a factor of N. The roots of this characteristic polynomial are the arguments of the Mittag-Leffler functions.

There are several benefits to this solution technique. The Mittag-Leffler solution is analogous in practice to the integer order solution. Consequently, closed form solutions can be written down easily using traditional techniques. For state formulations, the eigenvalues of the plant matrix are the arguments of the Mittag-Leffler functions. These eigenvalues are the "fractional poles" of the system and can be plotted in the complex plane. The complex plane containing these roots is a Riemann surface (7:303). Figure 1 shows the Riemann surface and an example mapping between the fractional and integer order spaces for β equal to 1/3:

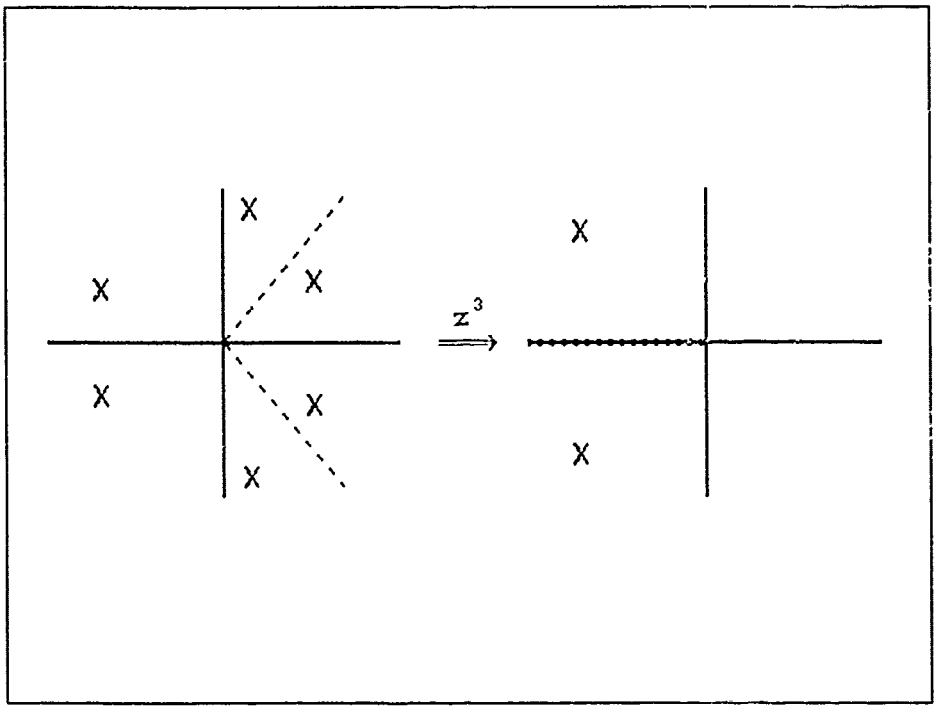


Figure 1 - Riemann Surface and Mapping to the Integer Complex Plane

To go from a fractional space to the integer space, the fractional space must be transformed by $z^{1/\beta}$. The wedge centered around the positive real axis on the Riemann surface which is $360° \beta$ wide is the principal branch of the transformation $\lambda=(\lambda_f)^{\beta}$ and is called the primary Riemann sheet. In Figure 1, the boundary of the primary Riemann sheet is at $\pm$ 60°. The corresponding boundary in the integer space is along the negative real axes. The roots on the primary Riemann sheet are the only roots mapped onto the integer order complex plane. The remaining roots map to other Riemann surfaces (not shown). The roots on the other surfaces determine much of the fractional behavior. The plane containing the entire Riemann surface will be referred to as the β-plane. For an integer order system posed in fractional equations, the roots on the β-plane are symmetric with respect to the boundaries between the different Riemann sheets. Portions of the Mittag-Leffler functions add out and leave the traditional exponential solutions. It will be shown later that the concept of the β-plane is useful from a design standpoint.

There are also some disadvantages inherent in Mittag-Leffler solution technique. The most obvious is that the Mittag-Leffler function is, by definition, an infinite sum. Consequently, using the definition directly in numerical calculations is impossible. Computation enforces truncating the series which can lead to convergence problems. Another difficulty from a design standpoint is the lack of understanding the transient behavior of Mittag-Leffler functions. This compounds the problem of determining the number of terms necessary in computation.

The Laplace Transform Solution

The Laplace transform has been shown by many (9:2047; 6:138,141-143; 12:247-275) to have analogous use in the representation of fractional order differential equations. The familiar differentiation property is still valid:

$$\mathcal{L}[D^{\alpha}[x(t)] = s^{\alpha}\mathcal{L}[x(t)], \qquad \alpha<1 \tag{10}$$

where

$$\mathcal{L}[x(t)] = X(s) = \int_{0}^{\infty} x(t)e^{-st}dt \tag{11}$$

The difficulty in this technique arises in the calculation of the inverse Laplace transform. Bagley and Torvik have shown the calculation of the inverse Laplace transform for the impulse response of fractional differential equations (6:141-143). The impulse response was solved for because it leads to most particular solutions of interest through convolution. The inverse Laplace transform is calculated using contour integration in the complex plane. The inverse Laplace transform of a function $X(s)$ is defined as:

$$x(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} X(s)\ e^{st}\ ds \tag{12}$$

The specific contours of the integration are shown in Figure 2. Contour 1 becomes the inversion integral as R approaches infinity. The result is the residues at the poles. Contours 2, 4 and 6 can be shown to be zero as R approaches infinity and ρ approaches zero (6:142). Contours 3 and 5 are the portion of the integration that are affected by fractional order equations. Since these integrals

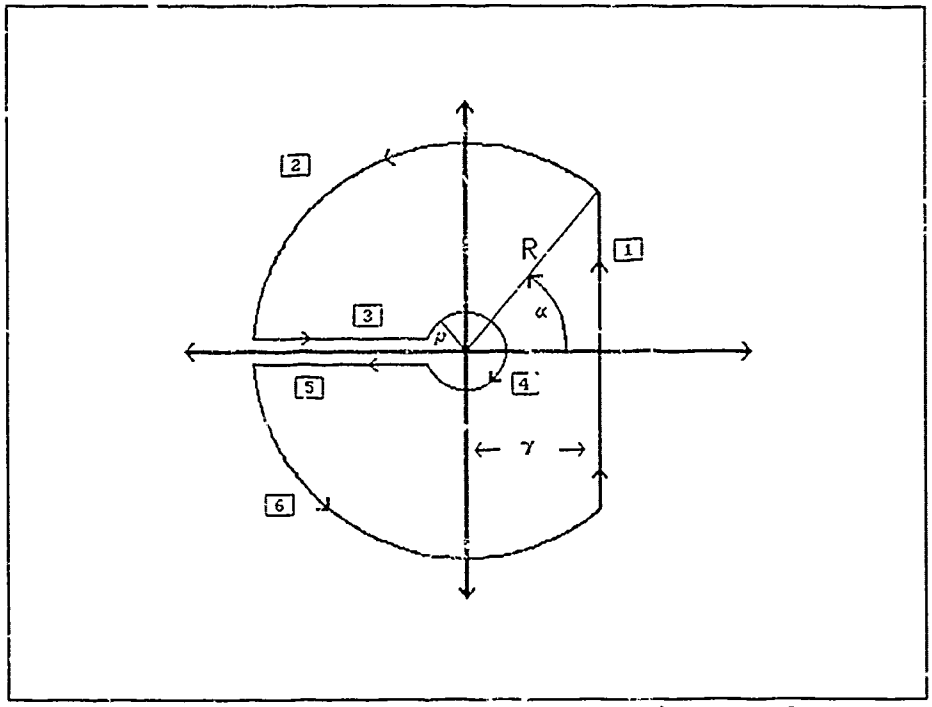


Figure 2 - Contours of Integration Used to Evaluate the Inverse Transform of the Impulse Response (6:141)

lie along the branch cut (that is. the negative real axis), their combined contribution is termed the branch cut integral. Since their directions are opposite. their contribution is their difference. For integer cases. the integration of contour 3 is equal to the integration of contour 5 and their difference is zero. For fractional cases, the integration of contour 3 is equal to the *conjugate* of the integration of contour 5. In general, they do not sum to zero.

The impulse response of a fractional order differential equation is:

$$x(t) = \sum_{k=1}^{n} C_k \cdot e^{i\omega_k t} + I(t) \qquad (13)$$

where

$I(t)$ ≡ branch cut integral

The residues can be found by the "limiting process" just as in the integer order case (6:142). Specifically,

$$C_k = \lim_{s \to i\omega_k} \{ (s - i\omega_k) \cdot X(s) \} \tag{14}$$

For a Laplace transform of the form

$$X(s) = \frac{1}{\sum_{j=0}^{n} K_j \cdot s^{\beta j}} \tag{15}$$

where

K_j = real constants

the branch cut integral can be shown to be

$$I(t) = \frac{1}{\pi} \int_0^{\infty} \left\{ \frac{\left[\sum_{j=0}^{n} K_j \cdot r^{\beta j} \cdot \sin(\beta j \pi) \right] \cdot e^{-rt}}{\left[\sum_{j=0}^{n} K_j \cdot r^{\beta j} \cdot \cos(\beta j \pi) \right]^2 + \left[\sum_{j=0}^{n} K_j \cdot r^{\beta j} \cdot \sin(\beta j \pi) \right]^2} \right\} dr \tag{16}$$

It is obvious that the evaluation of the branch cut integral is less than trivial. It should be noted that the branch cut integral is always bounded and stable due to the arguments of the exponential in the numerator of the integrand. It is also obvious from the exponential that the branch cut integral has a maximum magnitude at time equal to zero.

The Laplace transform technique does have some obvious advantages. The exponentials associated with the response are easily determined. It is these

exponentials that determine the stability of the system. Also, the exponentials allow for some understanding of the overall response of the structure. This technique sums all of the "fractional" behavior into the branch cut integral. The integral can be analyzed to determine which coefficients minimize or maximize its magnitude. Bagley has shown that the fractional order solution is continuous everywhere (1:73-76) (as could expected if it were modelling structural motion). Consequently, the initial value of the integral (which is also its maximum magnitude) is simply the negative of the sum of the residues for a system starting from rest. The primary disadvantage of the Laplace transform technique is that the evaluation of the integral itself leads to approximations when numerically implemented.

III. The Half Order Case

Up until now, the focus has been on fractional order differential equations in general. Now, the half order case will be looked at exclusively. This is due to some favorable features of the β equal to 1/2 case. Aspects from both of the solution techniques presented will be discussed.

For the half order case, the state vector will be twice as large as the integer order state vector. For modelling the dynamics of a structure, the integer order states are the position and velocity of the structure. Now, the 1/2 derivative between position and velocity will be sensed as well as the 3/2 derivative between velocity and acceleration. The model of a system is assumed an integer order representation. It must be posed in the fractional order equations to allow for the additional states being sensed to appear in the mathematics.

Given the model of a structure, it can always be put in the following form:

$$\hat{D}^{1/2}(\underline{x}) = A\underline{x} + B\underline{u} \tag{17}$$

where

$\underline{x}$ = state vector

$\underline{u}$ = input vector

A = plant matrix

B = control matrix

The above half order state equation will be the starting point for the control algorithm discussed in the next chapter.

The Half Order Mittag-Leffler

The half order Mittag-Leffler function is defined as:

$$E_{1/2}(x) \equiv \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + k/2)} \tag{18}$$

When the above equation is looked at carefully, it is seen that the sum *includes* the definition of an exponential. Indeed, any rational β of the form *1/N* would lead to an embedded exponential. What is left if the exponential is extracted is the half integral of an exponential. The definition of a fractional order integral can be found directly by integrating the definition of the fractional derivative one full time. The Riemann-Liouville δ-order fractional integral is defined as (5:307):

$$\hat{I}^{\delta}[x(t)] \equiv \int_0^t \frac{x(t-\tau)}{\Gamma(\delta)\ \tau^{1-\delta}} d\tau \tag{19}$$

It is a linear operator just like the fractional derivative. The half order Mittag-Leffler function can now be written as

$$E_{1/2}(\sigma t^{1/2}) = e^{\sigma^2 t} + \sigma \cdot \hat{I}^{1/2}\left[e^{\sigma^2 t}\right] \tag{20}$$

where

σ = constant (real, imaginary or complex)

It should be noted that if σ above is replaced with $-\sigma$, the only change in the above equation is the sign of the half integral. To investigate the stability of the half order Mittag-Leffler function, the half order integral of an exponential must be characterized.

Applying the definition from Eq (19) to an exponential yields:

$$\hat{I}^{1/2}[e^{\sigma^2 t}] = \frac{e^{\sigma^2 t}}{\Gamma(1/2)} \int_0^t \frac{e^{-\sigma^2 \tau}}{\tau^{1/2}} d\tau \tag{21}$$

The above equation is a disguised form of the incomplete gamma function. The incomplete gamma function of 1/2 is:

$$\gamma(1/2, t) \equiv \int_0^t \frac{e^{-\tau}}{\tau^{1/2}} d\tau \tag{22}$$

Consider the change of variables

$$\tau = \sigma^2 \eta \qquad d\tau = \sigma^2 d\eta \tag{23}$$

The definition can now be written as

$$\gamma(1/2, t) = \sqrt{\sigma^2} \int_0^{t\sigma^2} \frac{e^{-\sigma^2 \eta}}{\eta^{1/2}} d\eta \tag{24}$$

The constant brought outside of the integral can not be simplified without changing the meaning of the expression. If t is now allowed to approach infinity to determine asymptotic behavior, the following is the result:

$$\frac{\gamma(1/2, \infty)}{\sqrt{\sigma^2}} \equiv \frac{\Gamma(1/2)}{\sqrt{\sigma^2}} = \int_0^{\infty} \frac{e^{-\sigma^2 \eta}}{\eta^{1/2}} d\eta \tag{25}$$

Combining Eqs (20), (21) and (25), it is now seen that for large t,

$$E_{1/2}(\sigma t^{1/2}) \sim e^{\sigma^2 t} + \frac{\sigma}{\sqrt{\sigma^2}} e^{\sigma^2 t} \tag{26}$$

The above expression contains all of the stability information. It shows that for large t, any σ in the right half side of the complex plane leads to the sum of two

identical exponentials while any σ in the left half side leads to their difference. Therefore, all arguments of the Mittag-Leffler functions that have negative real parts are stable. The arguments with a positive real part that are smaller than the magnitude of their imaginary part will yield stable solutions since the square of the argument has a negative real part.

For any physical system, the Mittag-Leffler functions will appear in conjugate pairs analogous to exponentials for integer order systems. Also, the coefficients on conjugate pairs will themselves be conjugates. Expanding conjugate Mittag-Leffler functions yield

$$\begin{aligned}(a + bi)\,E_{1/2}[(c + di)\,t^{1/2}] + (a - bi)\,E_{1/2}[(c - di)\,t^{1/2}] = \\ 2\,e^{(c^2 - d^2)\,t}\cdot[a\cdot\cos(2cdt) - b\cdot\sin(2cdt)] + \\ (a + bi)\,(c + di)\cdot\hat{I}^{1/2}[e^{(c + di)^2 t}] + \\ (a - bi)\,(c - di)\cdot\hat{I}^{1/2}[e^{(c - di)^2 t}]\end{aligned} \tag{27}$$

where

a,b,c,d = real numbers

The importance of the above equation is numeric in nature. The exponentials have been extracted and consequently can be calculated exactly. In this manner, only half of the solution has to truncate an infinite sum.

As previously mentioned, the primary Riemann sheet is a wedge centered around the positive real axis in the β-plane which is $360°\,\beta$. For half order systems, the primary Riemann sheet is the *entire* right half side of the plane. Only half of the Mittag-Leffler functions for a given system will have their arguments on the primary sheet. The significance of this will be explored later.

The Half Order Branch Cut Integral

The Laplace transform method yields the residues and the branch cut integral in a straight forward manner. The poles are found using the "limiting process" and the branch cut integral can be simplified from Eq (16). For a system of the form

$$\ddot{x} + aD^{3/2}(x) + b\dot{x} + cD^{1/2}(x) + dx = f(t) \qquad (28)$$

where

a,b,c,d = real numbers

the contribution of the branch cut is

$$I(t) = \frac{1}{\pi}\int_0^{\infty}\left\{\frac{[cr^{1/2} - ar^{3/2}]\,e^{-rt}}{[r^2 - br + d]^2 - [ar^{3/2} - cr^{1/2}]^2}\right\} dr \qquad (29)$$

From the above expression it is obvious that for a and c identically zero, the branch cut integral is zero. For a given integer order system model, the smaller the fractional gains are, the smaller the contribution of the branch cut integral (as expected). As previously mentioned, the magnitude of the branch cut integral is largest at time zero because of the decaying exponential in the numerator of the integrand. The denominator of the integrand will become small as the integration variable r passes by the roots of r^2-br+d. Consequently, the integral will tend to get large - especially if their are positive real roots. The negative sign only changes the sign of the roots of r^2+br+d which is the original form of the equation with a and c identically zero. Therefore, given the coefficients of the fractional order terms, the branch cut integral will be minimized if the original system is

purely oscillatory. If the roots of the original system are real, the fractional order terms will have a dramatic effect on the solution.

Hybrid Analysis

The two solution techniques can be used in parallel to better understand the nature of the fractional order solutions. The Mittag-Leffler solution posed in the β-plane is elegant but does not easily lend itself to interpretation. The expansion of the Mittag-Leffler function will produce an exponential, but it is not necessarily the pole of the system. Consequently, the relevancy of the pole structure is not immediately obvious. The Laplace transform solution, however, explicitly separates the fractional nature of the response from the integer order exponential solution. Therefore, the poles of the system will be known and can be identified.

If the β-plane and the pole structure on it undergo the transformation z^2, the system will be posed in integer space. Any pole on the primary sheet will be squared and placed on the integer order complex plane. The other poles will be placed on other sheets of the Riemann surface. The poles on the primary Riemann sheet are the actual system poles while the poles on the other branches are accounted for in the branch cut integral.

In the integer complex plane, any pole in the right half side of the plane causes instability while any pole in the left half side is stable. An equivalent statement is any pole with its phase magnitude between 0° and 90° is unstable while poles with their phase magnitudes between 90° and 180° are stable. When a

complex number is squared, the magnitude is squared while the phase is doubled. Therefore, *roots of a half order equation on the primary Riemann sheet with a phase magnitude less than 45° are unstable while roots with phase magnitudes between 45° and 90° are stable.* For fractional orders other than 1/2, the same type of phase magnitude analysis determines stability.

The above conclusion is outlined by the previous stability analysis of Mittag-Leffler functions and the expression for the branch cut integral. The Laplace transform analysis showed that the poles of the system determine the stability while the fractional behavior contained in the branch cut integral is always stable. This shows that the poles in the β-plane on the primary sheet determine stability while those off of it are *always* stable. This is restating the 45° stability criteria in the β-plane demonstrated in the analysis of the half order Mittag-Leffler function. It was shown that half integrals negate the growing exponentials associated with the non-principal roots of the system as time approaches infinity. In fact, *the fractional behavior is simply the difference between the exponentials associated with non-principal roots and the sum of all the half integrals associated with the system.* Armed with a better understanding of fractional systems, a control algorithm can be formulated.

IV. Pole Placement in the β-plane

An understanding of the fractional order behavior enables the design of a control scheme posed in the fractional order notation. Design in the β-plane is now possible since the relevancy of the pole structure is known. A pole placement algorithm can be devised to select the position of all of the roots of the system. Although only the β equal to 1/2 case will be analyzed, the results will generalize.

Restating Eq (17) for convenience, the model of the structure is in the following form:

$$\hat{D}^{1/2}(\underline{x}) = A\underline{x} + B\underline{u} \tag{30}$$

where

$\underline{x}$ = state vector

$\underline{u}$ = input vector

A = plant matrix

B = control matrix

It should be noted that the dimension of the input vector is equal to the number of actuators on the structure. To specify the control law, the input vector will now be defined as:

$$\underline{u} = -K\underline{x} \tag{31}$$

where

K = gain matrix

This is full state feedback of the system states. The number of rows in the matrix K is equal to the number of actuators on the structure. For multiple actuators, the solution is not unique for fractional or integer order systems. External inputs can be ignored without loss of generality since the closed loop pole structure will determine system behavior. The equation that determines the β-plane structure is

$$\hat{D}^{1/2}(\underline{X}) = F\underline{X} \tag{32}$$

where

$$F \equiv A - BK$$

The individual elements of K can be chosen to place the roots of the system at desired locations. Pole placement on the β-plane is the same algorithm as pole placement on the integer complex plane. However, increasing the complexity of the system by adding twice as many gains is pointless unless there is a benefit to be realized.

Although all of the eigenvalues must be specified to *uniquely* determine the solution, some eigenvalues have more impact on the solution than others. The fractional formulation shows that regardless of the position of the roots on the left half side of the β-plane, the solution is always stable. *Only the eigenvalues on the primary Riemann sheet must be specified to determine system stability.* Consequently, the poles on the right half side of the β-plane can be chosen to be stable while the poles on the left hand side can be allowed to vary. This will

affect the transient behavior of the system by introducing additional damping. The additional damping is the result of energy decaying out of the system proportional to the fractional states and not just the velocity of the structure.

The damping added by fractional feedback is commendable for the application of motion suppression but may sometimes be modest. A system warranting active damping will by oscillatory in nature. The previous analysis of the branch cut integral showed that an oscillatory system gives rise to a smaller contribution than a heavily damped system. The actual size of the contribution is problem specific. Regardless if it is small or not, the solution still has a very favorable quality. Permitting the roots to vary on the non-principal sheet allows for unprecedented flexibility in the pole placement algorithm.

Let the desired integer order poles of the system be:

$$P_I = \{ p_{I,1}, \; p_{I,2}, \; p_{I,3}, \; \ldots \; p_{I,n} \} \tag{33}$$

where

P_I = set of desired poles on complex plane

The transformation of Eq (9) is used to determine the pole location on the β-plane. Specifically,

$$p_j = [p_{I,j}]^{1/\alpha} \tag{34}$$

Define

$$\Omega(r) \equiv (r - p_1)(r - p_2)(r - p_3) \ldots (r - p_n) \tag{35}$$

where

$\Omega(r) \equiv$ omega polynomial

The roots of this polynomial of order n are the desired poles on the primary Riemann sheet. The characteristic equation for the fractional order system is a polynomial of order $2n$:

$$| rI - F | = 0 \tag{36}$$

where

$I =$ identity matrix

The above equation must contain the omega polynomial as a factor to guarantee the desired pole location on the primary Riemann sheet. Synthetic division of the omega polynomial into the characteristic equation yields a polynomial of order n with a remainder of order $n-1$:

$$\frac{| rI - F|}{\Omega(r)} = \Psi(r) + \frac{R(r)}{\Omega(r)} \tag{37}$$

where

$\Psi(r) \equiv$ psi polynomial

$R(r) \equiv$ remainder polynomial

It is evident that the psi and remainder polynomial are constrained to be zero. The psi polynomial contains the n non-principal roots of the system. The Routh-Hurwitz stability criterion (11:222-224) can be used on the coefficients of the psi polynomial to constrain the roots to remain in the left half side of the β-plane. The result will be n inequality constraints. The reason the β equal to 1/2 case was

chosen is now evident. It is the only fractional order that has the branch cut in the β-plane on the imaginary axis. These constraints would be more difficult to determine for other fractional orders. As already mentioned, the remainder polynomial is of order *n-1*. It must be zero for r equal to all n principal roots of the system. This is only possible if all n coefficients are identically zero. This will produce n equality constraints.

In summary, the fractional order pole placement algorithm situates the roots on the primary Riemann sheet at their desired locations while allowing the remaining roots to vary on the non-principal sheet. Equality and inequality constraints between the gains are generated. This added flexibility may allow decreased sensitivity to errors.

V. Optimization for Robustness

The previous chapter discussed pole placement in the β-plane and concluded with an algorithm that produced constraints. The next logical step is to optimize the algorithm for an advantageous quality. The most redeeming attribute a controller can possess is an insensitivity to errors in the system. This insensitivity to errors is a property called robustness. Although robustness is easily understood in principle, a mathematical expression measuring it accurately can be less than trivial. The object is to determine if the added latitude of the fractional pole placement technique can be used to make the poles on the primary Riemann sheet more robust.

Condition Numbers

A beneficial property of the Mittag-Leffler solution to fractional order equations is the conventional matrix notation. This allows linear system theory to be applicable. One method to characterize the sensitivity of eigenvalues in a matrix is determining the condition number associated with the matrix of eigenvectors. The condition number associated with the eigenvector matrix is defined as (10:1131):

$$c = \|X\| \, \|X^{-1}\| \geq 1 \tag{38}$$

where

$c \equiv$ condition number

X = matrix of eigenvectors

The condition number is a measure of how singular or ill conditioned an eigenvector matrix is. Alternatively, it is also measuring the "amount of orthogonality" between the eigenvectors of the system (10:1141). The more orthogonal the eigenvectors of the solution are, the better. A robust system has solutions that are as decoupled as possible so that an error in one mode wil' primarily affect only itself. A high condition number characterizes an ill conditioned matrix which will magnify errors. A condition number of one denotes a perfectly conditioned matrix which will minimize the effect of errors. Only a normal matrix is perfectly conditioned since it has orthogonal eigenvectors.

For application to the pole placement algorithm, the eigenvalues do not need to be equally robust. The eigenvalues on the primary Riemann sheet are the only ones that determine stability *if* the others are constrained to remain on their own sheet. Traditional optimization of the above definition for the condition number should not be used since half of the eigenvalues do not need to be optimized. The sensitivity of *specific* eigenvalues needs to be measured mathematically. To accomplish this, the asymmetric eigenvalue problem must be exploited. The asymmetric eigenvalue problem consists of two eigenvectors for each eigenvalue:

$$F\underline{v_k} = \lambda_k \underline{v_k} \qquad \underline{y_k^T}F = \lambda_k \underline{y_k^T} \tag{39}$$

where

$$F \equiv A - BK$$

λ_k = k^{th} eigenvalue

$\underline{v}_k$ = right eigenvector of λ_k

$\underline{w}_k$ = left eigenvector of λ_k

The right eigenvector is the traditional eigenvector of the solution. The two eigenvectors must now be distinguished between so there is no confusion. The condition number of a *specific eigenvalue* is defined as (10:1131):

$$c_k = \frac{\|\underline{w}_k\|_2 \ \|\underline{v}_k\|_2}{|\underline{w}_k \cdot \underline{v}_k|} \geq 1 \qquad (40)$$

where

c_k = condition number of k^{th} eigenvalue

$\underline{v}_k$ = right eigenvector of λ_k

$\underline{w}_k$ = left eigenvector of λ_k

Although the above does not look like Eq (38) at first glance, the two definitions are very similar. If each right eigenvector in the system is normalized to unity length, the inverse of the right eigenvector matrix is equal to the transpose of the left eigenvector matrix (10:1140). The relevancy of the magnitude of the above condition number is the same as for the previous condition number. A low condition number denotes low sensitivity to errors. The first definition did not distinguish between eigenvalues in its measure of robustness. Consequently, each eigenvalue was weighted equally. The above definition will allow each eigenvalue to be weighted separately or not at all.

The advantage of the above expression is that the robustness associated with an eigenvalue is written in terms of its own eigenvectors. Forcing all of the

right eigenvectors of a system to being orthogonal is *exactly* the same as forcing the left and right eigenvectors of each eigenvalue to being identical. For real eigenvalues, the condition number is simply the reciprocal of the cosine of the angle between the left and right eigenvectors. It should be noted that a general expression for an eigenvector will contain the associated eigenvalue in it. Consequently, the sensitivity of an eigenvalue is related to the eigenvalue itself.

The Cost Function

The cost function in an optimization algorithm is the expression that will be minimized or maximized subject to possible constraints. A cost function should accurately represent the property it measures while remaining as elementary as mathematically possible. For this application, the eigenvalue condition numbers can be modified to facilitate computation. The numerator and denominator of Eq (40) both calculate the magnitude of complex expressions. In practice, this leads to taking the square root of entire complex polynomials. If Eq (40) is squared, this can be avoided without losing the significance of the condition number. It should also be noted that the condition numbers of conjugate eigenvalues are identical. Consequently, only one eigenvalue per mode must appear in the cost function.

The cost function will be subject to the equality and inequality constraints derived in the pole placement algorithm. The constraints can either be appended to the cost function using the method of Lagrange multipliers or substituted into the expression to solve for only independent variables. The left and right

eigenvectors for a given eigenvalue can be written symbolically in terms of the gains, but only half of the gains are independent. Considering the large number of dependant gains in the problem, using redundant variables would prove to be burdensome. The equality constraints can be solved for half of the gains and substituted directly into the condition numbers. In conclusion, the cost function can be written as the sum of the squares of the condition numbers associated with each mode on the primary Riemann sheet:

$$J = \sum_{k=1}^{n/4} c_k^2 \tag{41}$$

where

J = cost function

n = order of fractional order plant matrix

c_k = condition number associated with the k^{th} mode

Controllable Canonical Form

It has already been mentioned that some state formulations are better to work with than others. One such representation is the controllable canonical form. It will be used exclusively in the remainder of this investigation due to one of its favorable properties. The right eigenvectors of the system can be written in a general form that facilitates calculation. This form is expressed in the following partitioned matrix. For a system with κ number of masses.

$$A = \begin{bmatrix} 0 & I \\ M & C_s \end{bmatrix} \tag{42}$$

where

A = plant matrix (from state equation)

0 = zero matrix (n-κ by κ)

I = identity matrix (n-κ by n-κ)

M = mass matrix ($N\kappa$ by $N\kappa$)

C_s = stiffness matrix ($N\kappa$ by $N\kappa$)

The matrix B also has a standard form. It contains all ones in the bottom row and zeros everywhere else. For a system expressed in controllable canonical form, the right eigenvector can easily be written as a function of the eigenvalue and mode shape. Specifically,

$$\underline{v} = [\phi_1 \ \phi_2 \ \ldots \ \phi_\kappa \ \lambda\phi_1 \ \lambda\phi_2 \ldots \ \lambda\phi_\kappa \lambda^2\phi_1 \ \lambda^2\phi_2 \ \ldots \lambda^2\phi_\kappa \ \ldots\ldots \ \lambda^{\kappa-1}\phi_1 \ \lambda^{\kappa-1}\phi_2 \ \ldots \ \lambda^{\kappa-1}\phi_\kappa]^T \tag{43}$$

where

λ = eigenvalue

$\underline{v}$ = right eigenvector associated with λ

ϕ_K = structural mode shape of mass κ

To appreciate this, consider the right eigenvector problem associated with an eigenvalue λ:

$$(A - \lambda I)\underline{v} = \underline{0} \qquad (44)$$

When calculating the elements of $\underline{v}$, the top n-κ rows of the matrix A-λI relate all of the components of $\underline{v}$ to the first κ components of the vector. This is only because the system model is in controllable canonical form. The first κ components are termed the structural mode shapes. The relationship between the mode shapes can be found from the remaining bottom rows of the matrix A-λI. This means that the open loop right eigenvectors can all be written compactly as a function of their own eigenvalue.

When feedback is applied, the relationship between the modes becomes more complicated. The expression for the right eigenvector can still be written as shown in Eq (43), but the relationships between the modes will in general change. The relationships are determined from any κ-1 rows of the bottom κ of the closed loop matrix F. This result yields a special case for the single actuator application.

When only a single actuator is used, a single row of gains will appear in one of the bottom κ rows of the matrix F. Since only κ-1 of the bottom κ rows of matrix F are necessary to determine the eigenvector, one of the rows yields redundant information. This means that the relationship between the structural modes *can not* change when only one actuator is employed. The only "variable" in Eq (43) will be the eigenvalue itself. For an integer order system, the eigenvalues are unique and therefore the eigenvectors will be also. Indeed, this is the reason robustness is not an issue in single actuator integer order cases. For the fractional

order system, the eigenvalues on the non-principal sheet are variables that appear in the optimization.

The benefit realized from this is in the calculation of the cost function for a structure with only one actuator. If a closed loop right eigenvector is only a function of the eigenvalue, then altering the gains within the constraints will not change the right eigenvector. This means that the 2-norm of the right eigenvector is a constant and therefore does not affect the optimization. For systems where the right eigenvectors are only functions of the eigenvalues, the cost function can be simplified:

$$J = \sum_{k=1}^{n/4} \frac{\|\underline{y}_k\|_2^2}{|\underline{y}_k \cdot \underline{v}_k|^2} \qquad (45)$$

The cost function must be minimized to produce the optimal solution. A gradient search technique can be implemented numerically to determine the gains that will minimize the cost function. The remaining dependant gains can be determined from the equality constraints. If the optimal solution violates the inequality constraints, the constraints must be applied separately and in combination to determine the true optimal solution. However, violating the inequality constraints will not always be harmful for the application of structural damping.

For structural damping, the exact transient behavior is not as crucial as long as the system is stable and damped. If a root of the system is optimized without applying the inequality constraints and migrates to a *stable* portion of the

primary Riemann sheet, the transient behavior will be altered but not detrimental a priori. If the additional pole in the system were lightly damped, the resulting behavior would not be desirable. If, however, the added pole increased damping or did not change the damping, the new configuration would be better since it is optimally robust. In actual practice, it will be simpler to determine only the equality constraints and optimize subject to them. If the result is unfavorable, then the inequality constraints can be determined and applied.

VI. Comparison of Fractional and Integer Order Solutions

The emphasis so far has been on optimizing the fractional pole placement algorithm. The focus of this chapter is to compare the results of the algorithm with traditional results. For the fractional order controller to have extraordinary value, it should be superior to a traditional controller on the same structure. For an integer order controller with more than one actuator, the solution must also be optimized. This is because the multiple actuator case does not yield unique results. An authentic comparison between fractional and integer order solutions must compare optimized results from both methods. Consequently, methods to compare the robustness between integer and fractional order solutions must be investigated.

Optimization Differences

There is a fundamental difference between the fractional and integer order optimization methods. For an integer order solution, each eigenvalue affects stability. Consequently, each eigenvalue has equal importance in the cost function. This is the reason that the cost function is traditionally the condition number associated with the eigenvector matrix (10:1131) as defined in Eq (38). In the fractional order controller, half of the eigenvalues are constrained variables. This difference allows the fractional order controller to optimize for robustness in an additional manner.

For the integer order optimization, all of the eigenvalues are specified. Robustness is achieved in a system by assigning the eigenvectors so that the system is as well conditioned as possible. As was previously mentioned, this occurs when the right eigenvectors are as "orthogonal" as possible. For a single actuator, the eigenstructure is unique since the relationships between the structural modes can not change. Consequently, robustness is not an issue.

For the fractional order optimization, half of the eigenvalues are subject to the optimization algorithm. The eigenvectors of the eigenvalues on the primary Riemann sheet will be optimized just like in the integer case. The right eigenvectors will be placed as "orthogonal" as possible. The entire eigenstructure on the non-principal Riemann sheet, eigenvalues included, will then be chosen so that their right eigenvectors are as "orthogonal" as possible with respect to themselves and the eigenvectors on the principal sheet. For a single actuator, the fractional order case must still be optimized to determine the eigenvalues on the non-principal sheet that yield a robust eigenstructure. It should be noted that as with the integer order case, the single actuator fractional controller *can not* change the eigenvectors of the eigenvalues on the principal sheet.

Condition Number Analysis

The most logical basis for comparison of fractional and integer order systems is the eigenvalue condition numbers. These are the condition numbers appearing in the cost function of the fractional order optimization algorithm. The definition is recalled from the previous chapter for convenience:

$$c_k = \frac{\|\underline{y}_k\|_2 \ \|\underline{v}_k\|_2}{|\underline{y}_k \cdot \underline{v}_k|} \geq 1 \qquad (46)$$

where

c_k = condition number of k^{th} eigenvalue

$\underline{v}_k$ = right eigenvector of λ_k

$\underline{y}_k$ = left eigenvector of λ_k

At first glance, it seems legitimate that the integer order system should be posed in the expanded fractional space to compare results. This would allow direct comparison of specific condition numbers on the relevant eigenvalues. It should be noted, however, that the condition numbers of the integer system posed in fractional space are not the same as the condition numbers of the original system in integer space. This is caused by fractional states present in mathematical model that are absent in the actual integer order system. Consequently, comparing corresponding condition numbers in fractional space is not the correct method.

The next logical step is to compare the condition numbers of the fractional solution to their integer order counterparts in integer space. This would ensure the fractional states did not interfere in the integer order solution. It appears the fractional solution would be the more robust if its condition numbers were smaller than the corresponding condition numbers of the integer system in integer space. It should be noted, however, that errors in λ_k from perturbations $O(\epsilon)$ in the elements of a square matrix are (10:1131):

$$error \approx n c_k \epsilon \tag{47}$$

where

n = number of eigenvalues in matrix

The previous expression shows that the size of a matrix has a bearing on robustness. This seems intuitive since a larger matrix corresponds to a more complex system. This would appear to contradict the former assertion. Applying Eq (47) would dictate that the fractional order condition numbers must be half of the corresponding integer order ones to be of equal robustness. In actuality, the sensitivity of an eigenvalue is a function of the eigenvalue itself and not just through the relationship of the eigenvectors in the condition number. This relationship is not expressed in the above equation. If the fractional and integer order solutions posed in their natural space had the same eigenvalues, Eq (47) would dictate the true comparison. In actuality, the fractional order system has eigenvalues that are *square roots* of the integer order eigenvalues. This complicates the comparison process. The true relationship appears impossible to generalize explicitly. The actual sensitivity will probably be related to the eigenvalue by an exponent. Regardless, it should be noted that if the fractional order condition numbers are less than half of the integer condition numbers the results are decisive in favor of the fractional controller. This, however, is a sufficient condition but not a necessary one. It is obvious that this comparison is not globally definitive. Nevertheless, it yields correct results when applicable.

Perturbation Analysis

The condition number analysis is only conclusive part of the time. Consequently, another comparison technique is required. A traditional perturbation analysis of the eigenvalues can be done to supplement the previous section. Deif (8:205-207) has performed a perturbation analysis for the symmetric eigenvalue problem that can be extended to the asymmetric case. The subscripts will be dropped since the following analysis is valid for any distinct eigenvalue and its associated eigenvectors. Assume for small ϵ that a perturbed system can be written as

$$\begin{aligned} F_p &= F + \epsilon F_1 + \epsilon^2 F_2 + \ldots \\ \lambda_p &= \lambda + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \ldots \\ \underline{V}_p &= \underline{V} + \epsilon\underline{V}_1 + \epsilon^2\underline{V}_2 + \ldots \\ \underline{Y}_p &= \underline{Y} + \epsilon\underline{Y}_1 + \epsilon^2\underline{Y}_2 + \ldots \end{aligned} \tag{48}$$

The right eigenvalue problem for a perturbed system can be written in terms of the above definitions. The zero order solution is simply the unperturbed eigenvalue problem. For a first order perturbation solution, all ϵ^2 terms or higher are disregarded. The zero order solution can be extracted from the first order solution and the remaining terms can have an ϵ factored out. The result is the following expression:

$$F\underline{V}_1 + F_1\underline{V} = \lambda\underline{V}_1 - \lambda_1\underline{V} \tag{49}$$

The above expression can now be pre-multiplied by the transpose of the left eigenvector. By definition, the first and third term are equal and can be removed.

The above expression can be solved for λ_1 and substituted back into the definition of λ_p. Therefore, the first order perturbation solution is

$$\lambda_p = \lambda + \epsilon \left[\frac{\underline{V}^T \cdot F_1 \cdot \underline{V}}{\underline{V}^T \cdot \underline{V}} \right] \qquad (50)$$

This solution is slightly different than the first order solution from the previous section. The condition number does not explicitly appear although the expression contains both eigenvectors.

The perturbation analysis has some strengths and shortcomings. One obvious benefit of the above approximation is that individual elements of the nominal matrix can be perturbed while other elements remain unchanged. This quality leads to the question of which elements should be perturbed. It is obvious that any element of the closed loop matrix F that contains a gain will be subject to errors. The sub-matrices M and C_s from the controllable canonical form will contain modelling errors whether gains are present or not. Perturbations in the main diagonal of the identity matrix could be caused by errors in the actual integrations of the acceleration signal. All of the elements remaining that are zero and not affected by a feedback gain will be immune to errors. This is because the zero elements arise from variable assignment in the model.

Another factor that must be considered is the magnitude of the perturbations. Each source of error has a different average amplitude. The error from acceleration integration will be very small. The errors in the gains of a controller will be larger than integration errors, but may still be rather modest. The modelling errors may be a magnitude greater than the gain errors. It will

depend on the specific application. These different error sources could demand different weightings for an authentic comparison. Since this type of analysis is problem specific, the choosing of error magnitudes could be tedious.

The best way to implement the perturbation equation is to perturb each element of the nominal matrix individually, determine the error in each eigenvalue from each perturbed element, and sum all of the magnitudes of the errors. This means that each element subject to perturbations in the closed loop matrix F must be perturbed individually. This will enable the determination of the magnitude of the error. This magnitude is the radius of a circle in the β-plane around the nominal eigenvalue. This circle will be referred to as the error circle. Given an individual perturbation ϵ on an element, the perturbed eigenvalue will be somewhere on the circumference of the error circle. The error radius is obviously proportional to ϵ since this is a first order perturbation solution. In reality, some of the errors will sum together and some will cancel out. For a given eigenvalue, summing the radii of the error circles from each perturbed element will give the absolute upper limit on the eigenvalue error. This maximum error would occur if each perturbation constructively added simultaneously. The controller producing the lowest limit on the sum of errors from all the eigenvalues on the principal Riemann sheet will be the more robust. The obvious disadvantage of this comparison technique is that it is *extremely* time consuming for even a simple system.

The condition number and perturbation analysis can both be used to compare results. The condition number analysis is inconclusive in many cases but

is simple to implement. When applicable it yields results quickly. The perturbation analysis is valid for all cases but is laborious to execute. Also, the desired weightings for each error source must be determined. This task could also be very burdensome if all of the error sources are modelled. The analysis could be simplified by only considering the larger sources of error. The outcome of these comparisons will dictate whether the fractional controllers are truly more robust.

VII. Example Problems

This chapter will present three example problems applying the techniques from the previous chapters. A model of the structure will be assumed. The damping of the closed loop response will then be specified. The damping of the actual response will be greater than the specified damping due to the fractional feedback. As previously mentioned, this additional damping will be small for a lightly damped structure. The pole placement algorithm will then be optimized for robustness. Lastly, the fractional and integer order solutions will be compared.

Although the following problems yield precise numeric solutions, a great deal can be learned from the general implications of the results. All of the problems presented will contain only one actuator. This does not skew the utility of the algorithm but rather emphasizes its forte. Specifically, any increase in robustness from the fractional order controller must come from the choice of eigenvalues on the non-principal sheet since the eigenstructure of the principal roots is fixed for both integer and fractional order controllers. Using only one actuator also allows the simplified version of the cost function to be applicable in the numerical optimization. For a structure with one actuator implementing full state feedback, the associated gains are contained in a vector which is the same length as the state vector. This drives the matrix B of the state space formulation to also being a vector.

The first two examples will model a structure as a single mass system. The last example will model the structure as a two mass system. For examples 1 and 2, consider the following model:

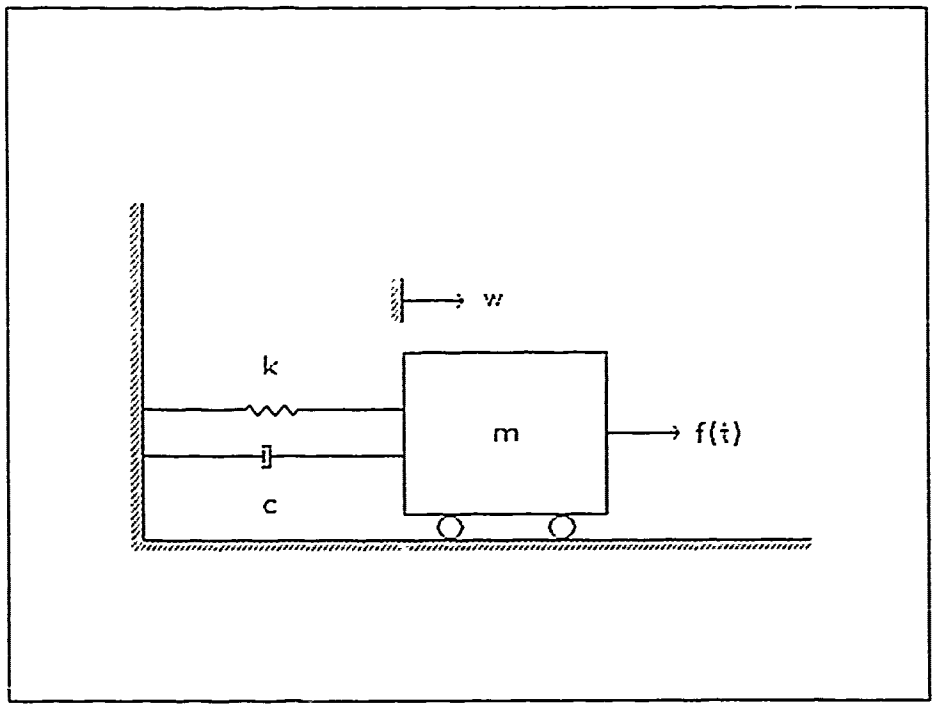


Figure 3 - Single Mass Model of Structure

The condition number associated with the mode of the mass will be subscripted with an m to avoid confusion with the viscous damping coefficient c.

Example Problem 1

Given:

$$m = 1 \text{ kg} \qquad k = 10 \text{ N/m} \qquad c = 2 \text{ N/m/s}$$

The differential equation associated with the open loop response is

$$\ddot{w} + 2\dot{w} + 10w = f(t) \tag{51}$$

The desired locations of the system poles will be:

$$P_I = \{-5+i,\ -5-i\} \tag{52}$$

It should be noted that these pole locations produce *heavy* damping.

Integer Order Solution

$$F_I = \begin{bmatrix} 0 & 1 \\ -10-K_1 & -2-K_2 \end{bmatrix} \qquad \underline{K_I^T} = [\ 16 \quad 8\] \qquad c_m = 13.5 \tag{53}$$

Fractional Order Solution

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10-K_1 & -K_2 & -2-K_3 & -K_4 \end{bmatrix} \tag{54}$$

The constraints on the gains from the pole placement algorithm are:

$$
\begin{aligned}
K_1 - (5.1)\,K_3 - (2.27)\,K_4 &= -24.787 \\
K_2 + (0.445)\,K_3 - (4.9)\,K_4 &= 3.56 \\
K_3 + (0.445)\,K_4 &> 2.9 \\
K_4 &> -0.445
\end{aligned}
\tag{55}
$$

As long as the equality constraints are satisfied, two of the eigenvalues of matrix F are equal to the square roots of the desired pole locations. The inequality constraints keep the non-specified eigenvalues off of the primary Riemann sheet. Optimizing the algorithm subject to the constraints yields:

$$
\underline{K}^T = [\ -9 \quad 0.3578 \quad 3.2563 \quad -0.3577\] \qquad c_m = 1.4571 \tag{56}
$$

It should be noted that the inequality constraints were never active.

Example Problem 2

Given:

$$m = 1 \text{ kg} \qquad k = 101 \text{ N/m} \qquad c = 2 \text{ N/m/s}$$

The differential equation associated with the open loop response is

$$\ddot{w} + 2\dot{w} + 101w = f(t) \tag{57}$$

The desired locations of the system poles will be:

$$P_I = \{-2+8i,\ -2-8i\} \tag{58}$$

This system will be moderately damped.

Integer Order Solution

$$F_I = \begin{bmatrix} 0 & 1 \\ -101-K_1 & -2-K_2 \end{bmatrix} \qquad \underline{K}_I^T = [\ -33 \quad 2\] \qquad c_m = 4.3125 \tag{59}$$

Fractional Order Solution

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -101-K_1 & -K_2 & -2-K_3 & -K_4 \end{bmatrix} \tag{60}$$

The constraints on the gains from the pole placement algorithm are:

$$
\begin{aligned}
K_1 - (8.25)\,K_3 - (29.15)\,K_4 &= -49.49 \\
K_2 + (3.535)\,K_3 + (4.25)\,K_4 &= 7.07 \\
K_3 + (3.535)\,K_4 &> -6.25 \\
K_4 &> -3.535
\end{aligned}
\qquad (61)
$$

The significance of the constraints is the same as before. Optimizing the algorithm for robustness subject to the constraints yields:

$$
\underline{K}^T = [\ -100 \quad 3.03 \quad 4.625 \quad -3.03\] \qquad c_m = 1.8606 \qquad (62)
$$

It should be noted that the inequality constraints were never active.

The two previous examples yield excellent results. The comparison of the examples with their integer order counter parts is straightforward since the condition number analysis is applicable. For *both* examples, the fractional order condition numbers were less than half of their corresponding integer order condition numbers. Therefore, the fractional order solutions are more robust in each case. As a side note, the condition numbers of the eigenvalues of the non-principal roots were the *exact same* as those for the principal roots in both examples. Although not explicitly calculated, the contribution of the branch cut integral will be much larger for Example Problem 1. This is due to the heavy damping desired for the closed loop response.

A trend can also be noted in the optimal solution for these single mode cases. Specifically, the position of the poles on the non-principal sheet followed the same general pattern. *The unspecified roots were placed at the negative reciprocal of the roots on the primary Riemann sheet.* This is no accident. The right eigenvector of a principal root is orthogonal to the right eigenvector of the negative reciprocal of the root. The associated conjugate eigenvectors are nearly orthogonal. Although not explicitly proven, this is a general result for a single mode case.

Example Problem 3

Consider the following model:

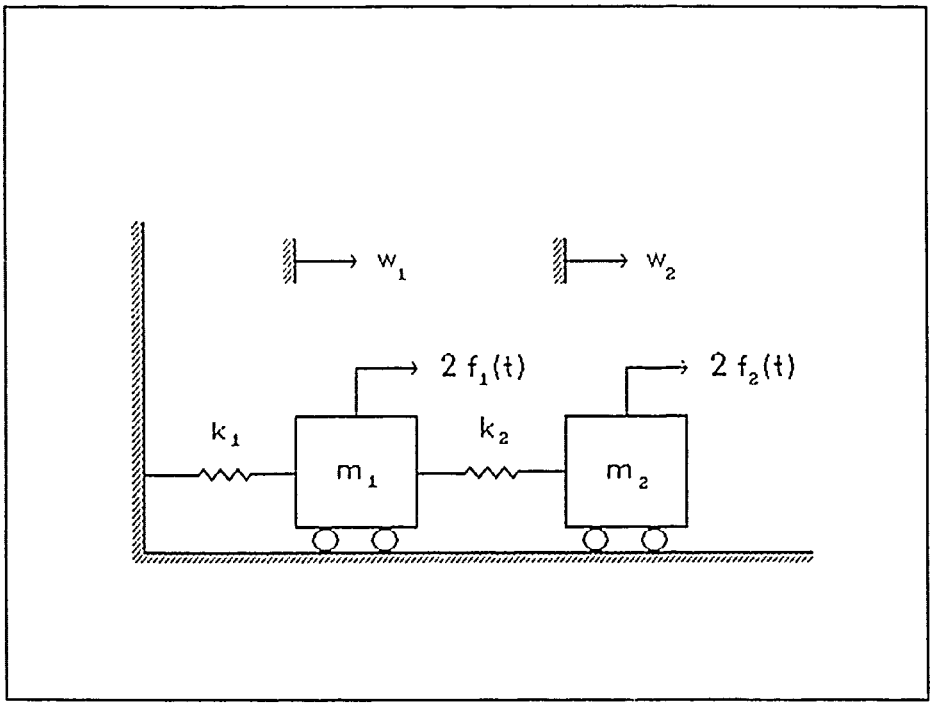


Figure 4 - Multi-Mass Model of Structure

Given:

$m_1 = 1$ kg $\qquad k_1 = 15$ N/m $\qquad f_1(t) = 0$

$m_2 = 2$ kg $\qquad k_2 = 18$ N/m

The differential equations associated with the open loop response are:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 2 \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{63}$$

It should be obvious that the above system is purely oscillatory. The desired damping will be 2% on each mode. The closed loop response will be *very* lightly damped. This yields the following:

$$P_I = \{ -0.0375+1.8724i,\ -0.0375-1.8724i,\ -0.124+6.203i,\ -0.124-6.203i \} \tag{64}$$

The branch cut integral for this example will be small.

Integer Order Solution

$$F_I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -33 & 18 & 0 & 0 \\ 9-K_1 & -9-K_2 & -K_3 & -K_4 \end{bmatrix} \tag{65}$$

$$\underline{K_I^T} = [\ -0.034\ \ 0.019\ \ -0.384\ \ 0.323\]$$

$$c_1 = 1.2769 \qquad c_2 = 3.2979 \tag{66}$$

Fractional Order Solution

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -33 & 18 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9-K_1 & -9-K_2 & -K_3 & -K_4 & -K_5 & -K_6 & -K_7 & -K_8 \end{bmatrix} \quad (67)$$

There are four constraints of equality and inequality each. In light of the last two examples, the equality constraints will be determined first and the optimal solution will be calculated. If the result yields eigenvalues that migrate across the branch cut *and* decreases damping, then the inequality constraints will be determined and applied. The equality constraints are:

$$\begin{aligned} K_1 - (5.48)K_5 + (1.20)K_6 - (18.92)K_7 + (4.48)K_8 &= 2.454 \\ K_2 + (2.20)K_5 - (2.60)K_6 + (9.33)K_7 - (6.31)K_8 &= -1.665 \\ K_3 + (3.27)K_5 - (0.35)K_6 + (5.54)K_7 - (0.71)K_8 &= -1.371 \\ K_4 - (0.82)K_5 + (2.13)K_6 - (2.25)K_7 + (2.22)K_8 &= -1.003 \end{aligned} \quad (68)$$

When the equality constraints are optimized, the result is:

$$\underline{K}^T = [\ -0.627\ -2.658\ -0.077\ \ 1.155\ \ 0.072\ \ 0.470\ -0.356\ -0.852\] \quad (69)$$

$$c_1 = 1.1777 \qquad c_2 = 4.7017$$

The inequality constraints do not need to be calculated because the optimized result yielded roots that remained on the non-principal Riemann sheet.

The roots on the non-principal Riemann sheet did not follow the trend from the single mass case. Two of the roots did, however, move toward the unit circle on the β-plane where their reciprocal eigenvalues were. The other two moved away from the integer order symmetric pattern. The pole assignment did not lend itself to any other interpretation. It did show that the optimal solution is not found from optimizing each mode separately. If each mode were optimized separately, all of the non-principal roots would be at negative reciprocal eigenvalues of the primary roots.

The results from Example Problem 3 are not conclusive using the condition number analysis. For the fractional solution, the first mode (the lower frequency mode) has a lower condition number than the integer solution while the second mode has a higher condition number. The fractional order condition number on the lower mode is not less than half of the integer order condition number. Therefore, a further analysis is required on the first mode. The fractional order condition number on the second mode is higher than the integer order condition number. It would appear that the second mode is more robust for the integer solution. Regardless, the results are inconclusive and the perturbation technique must be applied. As a side note, the condition numbers of the eigenvalues on the non-principal sheet were about the same as those that were optimized on the principal sheet.

It will be assumed for the perturbation analysis that the errors in the model of the structure are much greater than all of the other sources of error. This assumption is usually justified since the model is often determined

experimentally. Conversely, current integration techniques and gain accuracies are rather reliable. The stated assumption dictates that the perturbations will only be in the sub-matrices M and C_s of the plant model matrix A. Since only one error source is being modelled, the weighting will be unity on each element.

The perturbation analysis is a linearization around a nominal solution. Consequently, the error from a perturbation ϵ will be proportional to ϵ itself. As previously described, the analysis is best applied by perturbing each element of the sub-matrices M and C_s individually by ϵ and summing the maximum magnitude of the errors for each eigenvalue on the primary Riemann sheet. The result will be a radius R of maximum eigenvalue error. The results of applying the perturbation analysis are summarized below.

Integer Order Solution

Mode One: $R_1 = 0.529\ \epsilon$

Mode Two: $R_2 = 0.294\ \epsilon$

Fractional Order Solution

Mode One: $R_1 = 0.810\ \epsilon$

Mode Two: $R_2 = 0.230\ \epsilon$

The first mode appears more robust for the integer order controller while the second mode appears more robust for the fractional one. This example problem produced mixed results.

VIII. Closing Remarks

A control algorithm was derived that utilized fractional order states. The algorithm was best suited for active damping due to unusual transients inherent in fractional order systems. The algorithm was then optimized for robustness. Example problems were presented employing the derived techniques. The resulting solutions were then compared to the solutions from a traditional algorithm.

Conclusions

Control algorithms implementing fractional order feedback have a promising future. Only one fractional algorithm was considered in this investigation but the results were favorable. The fractional order controller was more robust than the integer order controller for the single mass cases. The same algorithm produced mixed results on the multi-mass example problem. It should be noted that only one multi-mass case was examined which makes decisive judgments difficult. Regardless, the conclusion is that feedback of fractional states can often produce a more robust system than traditional techniques can.

Recommendations

There are many possible avenues of future investigation. The first two example problems produced positive evidence that the fractional order controller has value. Although the third example problem had mixed results, there are still

multi-mass cases that may benefit globally from a fractional order controller. The most probable would be multi-mass systems which have inherent condition numbers much worse than the example did. The example was very well conditioned to begin with. On actual structures, the condition numbers can be in the hundreds or even thousands. The fractional order controller may prove superior in such cases.

There are some specific topics that need to be investigated in the future:

Cost Function

The cost function employed only optimized robustness for the principal roots of the system. It was *assumed* that the non-principal roots would remain on the non-principal sheet and consequently not affect stability. If, however, the non-principal roots are poorly conditioned, they could migrate to the unstable region of the principal sheet driven by errors. Consequently, a cost function accounting for the conditioning of the non-principal roots should be investigated to negate this possibility.

Furthermore, the condition number associated with an eigenvalue may not be the best measure of robustness. The condition number assumes possible error in every element of a given matrix. The perturbation analysis revealed that not every element in the closed loop matrix is subject to errors. Only certain elements will be subject to errors. Also, the condition number assumes unity weighting for

each error source. In reality, this will seldom be the case. A better cost function may be realized by using modified equations from the perturbation analysis.

Other Fractional Orders

The inequality constraints were not active in any of the example problems. Although not proven, it appears that the optimal solution does not tend to produce roots that migrate across the branch cut. If this is the case, other fractional orders should be investigated since the only reason the 1/2 order case was explored was so that the inequality constraints could easily be determined. There are an infinite number of other fractional orders that could be utilized. Some fractional orders may have some inherent benefits not perceived at present.

Other Fractional Control Algorithms

Only one type of fractional order controller was investigated during this study. There are a large number of other control schemes that could be modified to implement fractional feedback. Other algorithms may harness the additional information from the fractional states in a better or more efficient manner.

Case Studies

Only three examples were presented in the text. There are whole classes of examples not studied that may benefit from this algorithm. The class of problems with the most to gain would by systems that are inherently ill-conditioned.

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